

Half-BPS $SU(N)$ correlators in $\mathcal{N} = 4$ SYM

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ABSTRACT: In this note we study half-BPS operators in $\mathcal{N} = 4$ super Yang-Mills for gauge group $SU(N)$ at finite N . In particular we elaborate on the results of hep-th/0410236, providing an exact formula for the null basis operators algorithmically constructed there. For gauge groups $U(N)$ and $SU(N)$ we show that this basis is dual to the basis of multi-trace operators with respect to the two point function. We use this to extend the results of hep-th/0611290 concerning factorisation and probabilities from $U(N)$ to $SU(N)$. We also give a construction for a separate diagonal basis of the $SU(N)$ operators in terms of the higher Hamiltonians of the complex matrix model reduction of this sector.

KEYWORDS: AdS-CFT Correspondence, $1/N$ Expansion, M(atric) Theories.

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1. Introduction

The matching of the physics of the half-BPS sectors of $\mathcal{N} = 4$ super Yang-Mills in 4 dimensions and gravity in $AdS_5 \times S^5$ provides an important confirmation of the AdS/CFT correspondence [1]. Half-BPS operators are particularly tractable because their quantum numbers are not renormalised and certain extremal correlators are protected from renormalisation. This means that their correlators can be computed in the free theory, taken to strong coupling and then, via AdS/CFT, compared at large N with gravity [2].

In $\mathcal{N} = 4$ SYM half-BPS operators are built from traceless symmetric SO(6) tensor combinations of the six real scalars X_i , traced over their gauge indices (the X_i transform in the adjoint representation of the gauge group). We will be interested in the subset of those operators built from a single complex scalar $\Phi = X_1 + iX_2$, invariant under the remaining SO(4) subgroup of the SO(6) symmetry. The conformal structure of the theory dictates the correlator to be

$$\langle \Phi_a^\dagger(x) \Phi_b(y) \rangle = \frac{g_{ab}}{(x-y)^2} \tag{1.1}$$

where a, b run over the adjoint representation of the gauge group and g_{ab} is the inverse of the bilinear invariant form $g^{ab} = \text{tr}(T^a T^b)$. From now on we will drop the spacetime dependence of the correlators, because we are only interested in their group-theoretic structure.

For the U(N) gauge group the adjoint representation of the Lie algebra consists of N^2 $N \times N$ hermitian matrices. If we consider the matrix indices of $\Phi_j^i = \Phi_a (T^a)_j^i$, where T^a is an element of the adjoint representation of the Lie algebra of U(N), we find

$$\langle \Phi_j^i \Phi_l^k \rangle = g_{ab} (T^a)_j^i (T^b)_l^k = \delta_l^i \delta_j^k \tag{1.2}$$

The space of gauge-invariant chiral primary operators of a particular dimension in this SO(4)-invariant sector is made of products of traces (‘multi-traces’) of Φ . The number of fields Φ in the operator gives both the scaling dimension and the R-charge of the operator, which is a typical BPS saturation condition. In [3] the authors showed that linear combinations of the multi-trace operators called Schur polynomials diagonalise this two point function at finite N .

For dimension $k \ll N$ mixing between the trace operators is suppressed, so we map $\text{tr}(\Phi^k)$ to a graviton with angular momentum k around the the $X_1 - X_2$ plane of the sphere of $AdS_5 \times S^5$. If $k \sim N$ mixing between trace operators is no longer suppressed so we must look instead to the diagonal Schur polynomials for the appropriate objects on the gravity side. These correspond to D3 branes spinning in the geometry, called giant gravitons [4, 3]. As a complex matrix model the eigenvalues correspond to fermions in a harmonic potential [3, 5] and there is an exact map between the fermion distribution and the corresponding gravity solution with $\mathbb{R} \times \text{SO}(4) \times \text{SO}(4)$ symmetry [6].

For the $SU(N)$ gauge group elements of the Lie algebra are in addition traceless and the correlator receives a correction

$$\langle \Psi_j^{\dagger i} \Psi_l^k \rangle = \delta_l^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_l^k \tag{1.3}$$

Although at large N mixing between trace operators is still suppressed, at finite N this correction to the correlator complicates the combinatorics significantly. The Schur polynomials are no longer diagonal. In [7] a basis of the $SU(N)$ gauge-invariant operators called the null basis was found, which, while not diagonal, still has extremely nice properties, including a simple correlator. We will clarify the rôle of this basis here.

$U(N)$ is equivalent to $SU(N) \times U(1)$ up to a \mathbb{Z}_N identification. In the gauge theory the $U(1)$ vector multiplet is free, so the corresponding AdS field must decouple from all other fields living in the bulk, since gravity couples to everything. The field is a singleton field that lives at the boundary of AdS, corresponding to the centre of mass of the D3 branes [8].

In section 2 we will summarise the known $U(N)$ results and introduce the dual basis and its properties. Section 3 covers the corresponding $SU(N)$ picture, which is expanded upon in section 4 with detailed proofs. Section 5 extends the factorisation results of [9] from $U(N)$ to $SU(N)$ and section 6 describes the diagonalisation in terms of the higher Hamiltonians of the complex matrix model. There are some useful identities in section A.

2. $U(N)$ summary

For $U(N)$ theories the correlator for the complex scalar is

$$\langle \Phi_j^{\dagger i} \Phi_l^k \rangle = \delta_l^i \delta_j^k \tag{2.1}$$

We have three bases for the gauge invariant multi-trace polynomials of Φ .

1. The **trace basis**, of products of traces of Φ such as $\text{tr}(\Phi\Phi)\text{tr}(\Phi)$, is the obvious gauge-invariant basis. These multi-traces at level n are in one-to-one correspondence

with the $p(n)$ conjugacy classes¹ of the permutation group S_n where $p(n)$ is the number of partitions of n . Define a set of elements $\{\sigma_I\}$ in the permutation group S_n where each σ_I is an element of a different conjugacy class of S_n . All the possible multi-trace operators of dimension n are given by the $p(n)$ operators

$$\text{tr}(\sigma_I \Phi) = \sum_{j_1, j_2, \dots, j_n} \Phi_{j_{\sigma_I(1)}}^{j_1} \Phi_{j_{\sigma_I(2)}}^{j_2} \cdots \Phi_{j_{\sigma_I(n)}}^{j_n} \quad (2.2)$$

For example an element σ_I of S_5 made up of two 1-cycles and a 3-cycle, such as $\sigma_I = (1)(3)(245)$, gives an element of the trace basis $\text{tr}(\sigma_I \Phi) = \text{tr}(\Phi) \text{tr}(\Phi) \text{tr}(\Phi^3)$.

2. The **Schur polynomial basis** is defined as a sum of these trace operators over the elements σ of S_n , weighted by the characters of σ in the representation R of S_n

$$\chi_R(\Phi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \text{tr}(\sigma \Phi) \quad (2.3)$$

The representations R of S_n can be labelled by Young diagrams with n boxes, which also correspond to partitions of n . Thus there are $p(n)$ Schur polynomials of degree n . R also corresponds to a representation of $U(N)$.²

The correlation function of two Schur polynomials is diagonal for any value of N [3]

$$\langle \chi_R(\Phi^\dagger) \chi_S(\Phi) \rangle = \delta_{RS} f_R \quad (2.4)$$

f_R is computed by

$$f_R = \frac{n! \text{Dim}_R}{d_R} = \prod_{i,j} (N - i + j) \quad (2.5)$$

where Dim_R is the dimension of the $U(N)$ representation R and d_R is the dimension of the symmetric group S_n representation R . In the product expression we sum over the boxes of the Young diagram for R , i labelling the rows and j the columns.

We can invert the relation between traces and Schur polynomials using the identities in section A

$$\text{tr}(\sigma_I \Phi) = \sum_{R(n)} \chi_R(\sigma_I) \chi_R(\Phi) \quad (2.6)$$

where we sum over representations R of S_n with Young diagrams of n boxes. This gives us a compact formula for the correlation function of two elements of the trace basis

$$\langle \text{tr}(\sigma_I \Phi^\dagger) \text{tr}(\sigma_J \Phi) \rangle = \sum_R f_R \chi_R(\sigma_I) \chi_R(\sigma_J) \quad (2.7)$$

¹Conjugacy classes of S_n encode the different cycle structures of permutations.

²For a unitary matrix U the character of U in the representation R is given by $\chi_R(U)$ defined by this formula. That R is a representation of both S_n and $U(N)$ is a consequence of the fact that $U(N)$ and S_n have a commuting action on $V^{\otimes n}$, where V is the fundamental representation of $U(N)$.

3. Define the $p(n)$ elements of the **dual basis** by

$$\xi(\sigma_I, \Phi) := \frac{|\sigma_I|}{n!} \sum_{R(n)} \frac{1}{f_R} \chi_R(\sigma_I) \chi_R(\Phi) \quad (2.8)$$

where $|\sigma_I|$ is the size of the conjugacy class of σ_I . Note that $\xi(\sigma_I, \Phi)$ is constant on the conjugacy class of σ_I .

This basis is useful because it is dual to the trace basis using the inner product defined in (2.1), i.e.

$$\left\langle \xi(\sigma_I, \Phi^\dagger) \text{tr}(\sigma_J \Phi) \right\rangle = \delta_{IJ} \quad (2.9)$$

We can show this using the diagonality of the Schur polynomials (2.4) and the identity (A.1) in section A

$$\begin{aligned} \left\langle \xi(\sigma_I, \Phi^\dagger) \text{tr}(\sigma_J \Phi) \right\rangle &= \frac{|\sigma_I|}{n!} \sum_{R(n)} \frac{1}{f_R} \chi_R(\sigma_I) \sum_{S(n)} \chi_S(\sigma_J) \left\langle \chi_R(\Phi^\dagger) \chi_S(\Phi) \right\rangle \\ &= \frac{|\sigma_I|}{n!} \sum_{R(n)} \chi_R(\sigma_I) \chi_R(\sigma_J) \\ &= \delta_{IJ} \end{aligned} \quad (2.10)$$

The correlation function of two elements of the dual basis is given by

$$\left\langle \xi(\sigma_I, \Phi^\dagger) \xi(\sigma_J, \Phi) \right\rangle = \frac{|\sigma_I|}{n!} \frac{|\sigma_J|}{n!} \sum_R \frac{1}{f_R} \chi_R(\sigma_I) \chi_R(\sigma_J) \quad (2.11)$$

This matrix provides the change of basis from the trace basis to the dual basis

$$\sum_J \left\langle \xi(\sigma_I, \Phi^\dagger) \xi(\sigma_J, \Phi) \right\rangle \text{tr}(\sigma_J \Phi) = \xi(\sigma_I, \Phi^\dagger) \quad (2.12)$$

where we sum \sum_J over conjugacy classes of S_n . We have used identity (A.3) of section A. It follows that the matrix of correlators of the dual basis (2.11) is the inverse of the matrix of correlators of the trace basis (2.7)

$$\sum_J \left\langle \xi(\sigma_I, \Phi^\dagger) \xi(\sigma_J, \Phi) \right\rangle \left\langle \text{tr}(\sigma_J \Phi^\dagger) \text{tr}(\sigma_K \Phi) \right\rangle = \left\langle \xi(\sigma_I, \Phi^\dagger) \text{tr}(\sigma_K \Phi) \right\rangle = \delta_{IK} \quad (2.13)$$

In the large N limit we see from equation (2.5) that $f_R \rightarrow N^n$ so that the dual basis becomes, up to a factor, the trace basis

$$\xi(\sigma_I, \Phi) = \frac{|\sigma_I|}{n!} \sum_{R(n)} \frac{1}{f_R} \chi_R(\sigma_I) \chi_R(\Phi) \rightarrow \frac{|\sigma_I|}{N^n n!} \text{tr}(\sigma_I \Phi) \quad (2.14)$$

In this limit the duality of the two bases in equation (2.9) is just the well-known orthogonality of traces at large N .

3. $SU(N)$ summary

In $SU(N)$ our complex scalar is traceless. Denote the $SU(N)$ complex scalar by Ψ to distinguish it from the $U(N)$ complex scalar Φ which does have a trace. The correlator for Ψ is

$$\langle \Psi_j^{\dagger i} \Psi_l^k \rangle = \delta_l^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_l^k \quad (3.1)$$

We can relate this to the $U(N)$ correlator (2.1) by making the substitution $\Psi_j^i = \Phi_j^i - \delta_j^i \Phi_k^k / N$. If we feed this substitution into the $U(N)$ correlator we get the same result

$$\langle \Psi_j^{\dagger i} \Psi_l^k \rangle = \langle \left(\Phi_j^{\dagger i} - \delta_j^i \Phi_m^{\dagger m} / N \right) \left(\Phi_l^k - \delta_l^k \Phi_n^n / N \right) \rangle = \delta_l^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_l^k \quad (3.2)$$

This means that we can use *the same* correlator for both $U(N)$ and $SU(N)$, using operators built from Φ_j^i for $U(N)$ and from $\Psi_j^i = \Phi_j^i - \delta_j^i \Phi_k^k / N$ for $SU(N)$. This ability to move between the $SU(N)$ and $U(N)$ correlators using the substitution $\Psi_j^i = \Phi_j^i - \delta_j^i \Phi_k^k / N$ will be extremely useful in later formulae. In essence this substitution enforces the tracelessness condition.³

Ψ is traceless $\text{tr} \Psi = 0$ so we are going to need to consider elements of S_n without 1-cycles. Define C_n to be the subset of S_n with all the elements with 1-cycles removed. For example

- $C_1 = \emptyset$
- $C_2 = \{(12)\}$
- $C_3 = \{(123), (132)\}$
- $C_4 = \{[(12)(34)], [(1234)]\}$
- $C_5 = \{[(12)(345)], [(12345)]\}$

$[(12)(34)]$ means the conjugacy class of $(12)(34)$, which is $\{(12)(34), (13)(24), (14)(23)\}$.

Define a set of elements $\{\tau_i\}$ in C_n where each τ_i is an element of a different conjugacy class. There are $p(n) - p(n-1)$ conjugacy classes in C_n , since each element with a 1-cycle can be decomposed into a 1-cycle and an element of S_{n-1} .

The three bases of dimension n gauge-invariant polynomials of Ψ have some different properties to their $U(N)$ counterparts.

1. The **trace basis** is defined by the $p(n) - p(n-1)$ conjugacy classes of C_n

$$\text{tr}(\tau_i \Psi) \quad (3.3)$$

For $n = 2$ we have $\text{tr}(\Psi^2)$, for $n = 3$ we have $\text{tr}(\Psi^3)$, for $n = 4$ we have $\text{tr}(\Psi^2) \text{tr}(\Psi^2)$ and $\text{tr}(\Psi^4)$ and for $n = 5$ we have $\text{tr}(\Psi^2) \text{tr}(\Psi^3)$ and $\text{tr}(\Psi^5)$.

³Note that this method can also be applied to $O(N)$ and $\text{Sp}(2N)$. Elements of the Lie algebra of $O(N)$ are antisymmetric real matrices $\chi = -\chi^T$. We can obtain the $O(N)$ correlator by the substitution $\chi = i(X - X^T)$ where X is a hermitian generator of $U(N)$ (cf. [10]). Similarly for $\text{Sp}(2N)$ the real Lie algebra elements Π satisfy $J\Pi = (J\Pi)^T$ and their correlator can be found with $\Pi = J(X + X^T)$.

2. The $p(n)$ elements of the **Schur polynomial basis** $\chi_R(\Psi)$ are now neither independent nor diagonal. For each of the $p(n-1)$ Young diagrams T with $n-1$ boxes we have a linear relation between the Schur polynomials of dimension n

$$0 = \text{tr}(\Psi)\chi_T(\Psi) = \chi_{\square}(\Psi)\chi_T(\Psi) = \sum_{R(n)} g(\square, T; R)\chi_R(\Psi) \quad (3.4)$$

\square is the single box representation $\chi_{\square}(\Psi) = \text{tr}(\Psi) = 0$ and $g(\square, T; R)$ is the Littlewood-Richardson coefficient for compositions of representations. It is only non-zero if R is in $\square \otimes T$.

3. The **dual basis** is defined by the $p(n) - p(n-1)$ conjugacy classes of C_n

$$\xi(\tau_i, \Psi) := \frac{|\tau_i|}{n!} \sum_{R(n)} \frac{1}{f_R} \chi_R(\tau_i) \chi_R(\Psi) \quad (3.5)$$

It turns out that even for $SU(N)$ this basis is dual to the trace basis using the inner product defined in (3.1), i.e.

$$\left\langle \xi(\tau_i, \Psi^\dagger) \text{tr}(\tau_j \Psi) \right\rangle = \delta_{ij} \quad (3.6)$$

We will show that for $SU(N)$ this dual basis is exactly the null basis constructed algorithmically in [7].

The correlation function of two elements of the dual basis is given by

$$\left\langle \xi(\tau_i, \Psi^\dagger) \xi(\tau_j, \Psi) \right\rangle = \frac{|\tau_i|}{n!} \frac{|\tau_j|}{n!} \sum_R \frac{1}{f_R} \chi_R(\tau_i) \chi_R(\tau_j) \quad (3.7)$$

which is remarkably exactly the same as the $U(N)$ correlator of the dual basis (2.11), as proved in [7] for the null basis.

The matrix of correlators of the dual basis provides the change of basis from the trace basis to the dual

$$\sum_j \left\langle \xi(\tau_i, \Psi^\dagger) \xi(\tau_j, \Psi) \right\rangle \text{tr}(\tau_j \Psi) = \xi(\tau_i, \Psi^\dagger) \quad (3.8)$$

where we sum \sum_j over conjugacy classes of C_n . To get this result we can use the same argument as for the $U(N)$ case because we can add into the sum the remaining elements of S_n with 1-cycles, whose corresponding traces vanish. Thus the matrix of correlators of the dual basis is also the inverse of the matrix of correlators of the trace basis

$$\sum_j \left\langle \xi(\tau_i, \Psi^\dagger) \xi(\tau_j, \Psi) \right\rangle \left\langle \text{tr}(\tau_j \Psi^\dagger) \text{tr}(\tau_k \Psi) \right\rangle = \left\langle \xi(\tau_i, \Psi^\dagger) \text{tr}(\tau_k \Psi) \right\rangle = \delta_{ik} \quad (3.9)$$

where again we sum \sum_j over conjugacy classes of C_n .

4. $SU(N)$ details

Following [7] we define a derivative on the Schur polynomials of a general $N \times N$ matrix M_j^i by

$$D\chi_R(M) = \sum_{i=1}^M \frac{\partial}{\partial M_j^i} \chi_R(M) = \sum_{T(n-1)} g(\square, T; R) \frac{f_R}{f_T} \chi_T(M) \quad (4.1)$$

where we have given an exact formula for the derivative. We sum over representations T with $(n-1)$ boxes that differ from R by a ‘legal’ box. \square is the single-box fundamental representation; $g(\square, T; R)$ is a Littlewood-Richardson coefficient that is zero if R is not in $\square \otimes T$. The formula for the Littlewood-Richardson coefficient is given in section A. $\frac{f_R}{f_T}$ is the weight $(N-i+j)$ of the box removed from the Young diagram of R to get T , where i labels the row and j the column of the box in the Young diagram of R .

Using this we can Taylor expand for a constant k

$$\chi_R(M + k\mathbb{I}) = \sum_{F=0}^n \frac{1}{F!} k^F D^F \chi_R(M) \quad (4.2)$$

$$= \sum_{F=0}^n \frac{1}{F!} \sum_{T(n-F)} g(\square^F, T; R) \frac{f_R}{f_T} k^F \chi_T(M) \quad (4.3)$$

Here $g(\square^F, T; R) = g(\square, \dots, \square, T; R)$ with F \square 's. It counts the different legal ways we can build the representation R by adding F single-box representations \square to T . T has $(n-F)$ boxes. For example

$$g(\square^2, \square\square; \square\square\square) = 2 \quad (4.4)$$

because

$$\square \otimes \square \otimes \square\square = \square \otimes (\square\square \oplus \square\square) = 2 \square\square \oplus \square\square \oplus \square\square \oplus \square\square \quad (4.5)$$

We have therefore

$$\chi_R(\Psi) = \chi_R\left(\Phi - \frac{\text{tr } \Phi}{N} \mathbb{I}\right) = \sum_{F=0}^n \frac{1}{F!} \sum_{T(n-F)} g(\square^F, T; R) \frac{f_R}{f_T} \left(-\frac{\text{tr } \Phi}{N}\right)^F \chi_T(\Phi) \quad (4.6)$$

and conversely

$$\chi_R(\Phi) = \chi_R\left(\Psi + \frac{\text{tr } \Phi}{N} \mathbb{I}\right) = \sum_{F=0}^n \frac{1}{F!} \sum_{T(n-F)} g(\square^F, T; R) \frac{f_R}{f_T} \left(\frac{\text{tr } \Phi}{N}\right)^F \chi_T(\Psi) \quad (4.7)$$

These two equations are entirely compatible. If we feed the expression for $\chi_T(\Psi)$ given by (4.6) into (4.7) we recover $\chi_R(\Phi)$.

In [7] the authors algorithmically constructed a set of operators annihilated by the operator D which they called the null basis. Because they are annihilated by the operator D the Taylor expansion (4.2) is truncated to the $F=0$ terms.

Now we will show that that the $SU(N)$ dual basis $\xi(\tau_i, \Psi)$ for $\tau_i \in C_n$ given in (3.5) is indeed null and hence, using the substitution $\Psi = \Phi - \text{tr } \Phi/N$, we have

$$\xi(\tau_i, \Psi) = \xi(\tau_i, \Phi) \quad (4.8)$$

This is true because we get only the $F = 0$ terms in the Taylor expansion.

If we expand $\xi(\tau_i, \Psi)$

$$\begin{aligned} D\xi(\tau_i, \Psi) &= \frac{||[\tau_i]||}{n!} \sum_{R(n)} \frac{1}{f_R} \chi_R(\tau_i) D\chi_R(\Psi) \\ &= \frac{||[\tau_i]||}{n!} \sum_{R(n)} \chi_R(\tau_i) \sum_{T(n-1)} g(\square, T; R) \frac{1}{f_T} \left(-\frac{\text{tr } \Phi}{N} \right) \chi_T(\Phi) \end{aligned} \quad (4.9)$$

This looks monstrous but if we extract the sum over R and use the identity (A.4) for $g(\square, T; R)$ from section A, expanding it in characters of the symmetric group, we see that

$$\begin{aligned} \sum_{R(n)} \chi_R(\tau_i) g(\square, T; R) &= \sum_{R(n)} \chi_R(\tau_i) \frac{1}{(n-1)!} \sum_{\rho \in S_{n-1}} \chi_{\square}(\text{id}) \chi_T(\rho) \chi_R(\text{id} \circ \rho) \\ &= \frac{1}{(n-1)!} \sum_{\rho \in S_{n-1}} \chi_{\square}(\text{id}) \chi_T(\rho) \frac{n!}{||[\tau_i]||} \delta([\tau_i] = [\text{id} \circ \rho]) \end{aligned} \quad (4.10)$$

where we have used identity (A.1). Here id is the identity permutation made only of 1-cycles. But we know that τ_i has no 1-cycles so $[\tau_i] = [\text{id} \circ \rho]$ is never satisfied. Therefore the $SU(N)$ dual basis is indeed null $D\xi(\tau_i, \Psi) = 0$ and thus $\xi(\tau_i, \Psi) = \xi(\tau_i, \Phi)$ is true.

Note that this only works for the $SU(N)$ dual basis $\xi(\tau_i, \Psi)$ with $\tau_i \in C_n$. For a general $\sigma_I \in S_n$ with 1-cycles, $\sigma_I \notin C_n$, $\xi(\sigma_I, \Psi)$ is not null and we do not have $\xi(\sigma_I, \Psi) = \xi(\sigma_I, \Phi)$.

The correlator of two members of the $SU(N)$ dual basis (3.7) now follows very quickly because it must be the same as the $U(N)$ correlator

$$\left\langle \xi(\tau_i, \Psi^\dagger) \xi(\tau_j, \Psi) \right\rangle = \left\langle \xi(\tau_i, \Phi^\dagger) \xi(\tau_j, \Phi) \right\rangle = \frac{||[\tau_i]||}{n!} \frac{||[\tau_j]||}{n!} \sum_R \frac{1}{f_R} \chi_R(\tau_i) \chi_R(\tau_j) \quad (4.11)$$

Using

$$\left\langle \Psi^\dagger \text{tr } \Phi \right\rangle = 0 \quad \Rightarrow \quad \left\langle \Psi^\dagger \Psi \right\rangle = \left\langle \Psi^\dagger \Phi \right\rangle \quad (4.12)$$

we can also see that the duality of the multi-trace basis to the null basis follows from the $U(N)$ case

$$\left\langle \xi(\tau_i, \Psi^\dagger) \text{tr}(\tau_j \Psi) \right\rangle = \left\langle \xi(\tau_i, \Psi^\dagger) \text{tr}(\tau_j \Phi) \right\rangle = \left\langle \xi(\tau_i, \Phi^\dagger) \text{tr}(\tau_j \Phi) \right\rangle = \delta_{ij} \quad (4.13)$$

In the first equality we have used property (4.12) that $\langle \Psi^\dagger \Psi \rangle = \langle \Psi^\dagger \Phi \rangle$; in the second we have used property (4.8) that $\xi(\tau_i, \Psi) = \xi(\tau_i, \Phi)$; in the final inequality we have used the defining property of the $U(N)$ dual basis (2.9).

We would now like to show that the Schur polynomial basis is no longer diagonal for

SU(N). We can use (4.7) to see that

$$\begin{aligned}
\langle \chi_R(\Phi^\dagger)\chi_S(\Phi) \rangle &= \\
&= \sum_{F=0}^n \frac{1}{(F!)^2} \frac{1}{N^{2F}} \sum_{T(n-F)} \sum_{U(n-F)} g(\square^F, T; R)g(\square^F, U; S) \\
&\quad \times \frac{f_R f_S}{f_T f_U} \langle \chi_{\square^F}^F(\Phi^\dagger)\chi_{\square^F}^F(\Phi) \rangle \langle \chi_T(\Psi^\dagger)\chi_U(\Psi) \rangle \\
&= \sum_{F=0}^n \frac{1}{(F!)} \frac{1}{N^F} \sum_{T(n-F)} \sum_{U(n-F)} g(\square^F, T; R)g(\square^F, U; S) \frac{f_R f_S}{f_T f_U} \langle \chi_T(\Psi^\dagger)\chi_U(\Psi) \rangle \quad (4.14)
\end{aligned}$$

Separating out the $F = 0$ term and re-arranging we see that

$$\begin{aligned}
\langle \chi_R(\Psi^\dagger)\chi_S(\Psi) \rangle &= \langle \chi_R(\Phi^\dagger)\chi_S(\Phi) \rangle \\
&\quad - \sum_{F=1}^n \frac{1}{(F!)} \frac{1}{N^F} \sum_{T,U} g(\square^F, T; R)g(\square^F, U; S) \frac{f_R f_S}{f_T f_U} \langle \chi_T(\Psi^\dagger)\chi_U(\Psi) \rangle \quad (4.15)
\end{aligned}$$

which when applied recursively gives us

$$\begin{aligned}
\langle \chi_R(\Psi^\dagger)\chi_S(\Psi) \rangle &= \\
&= \sum_{F=0}^n \frac{1}{(F!)} \left(-\frac{1}{N}\right)^F \sum_{T,U} g(\square^F, T; R)g(\square^F, U; S) \frac{f_R f_S}{f_T f_U} \langle \chi_T(\Phi^\dagger)\chi_U(\Phi) \rangle \\
&= \sum_{F=0}^n \frac{1}{(F!)} \left(-\frac{1}{N}\right)^F \sum_T g(\square^F, T; R)g(\square^F, T; S) \frac{f_R f_S}{f_T} \quad (4.16)
\end{aligned}$$

This agrees with the calculation in equation (10.7) of [11] if we make the identification $g(\square^F, T; R) = \sum_U d_U g(U, T; R)$. This identification follows from the identities in section A and the fact that $d_U = \chi_U(\text{id}^{\circ F})$. The formula also agrees with the results from [7].

5. Factorisation and probabilities for SU(N)

Given that we have a basis and its dual we can write down factorisation equations for SU(N) correlators analogous to those described in [9] for U(N) correlators. For a conformal field theory like $\mathcal{N} = 4$ super Yang-Mills these factorisation equations let us write correlators on 4-dimensional surfaces with non-trivial topology in terms of correlators on the 4-sphere, just like factorisation of correlators on Riemann surfaces in two dimensions. Because of positivity properties of the summands in the factorisation equations we can interpret these summands as well-defined probabilities for a large class of processes. Since we are only interested in the combinatorics we will drop the spacetime dependences and any extraneous modular parameters.

If a complete basis for the local operators of our SU(N) theory is given by $\{\mathcal{O}_a\}$ and the metric on this basis from the two point function has an inverse G^{ab} , then for local

operators \mathcal{A} , \mathcal{B} the sphere factorisation is given by a sum of positive quantities [9]

$$\begin{aligned}
 \langle \mathcal{A}^\dagger \mathcal{B} \rangle &= \sum_{a,b} G^{ab} \langle \mathcal{A}^\dagger \mathcal{O}_a \rangle \langle \mathcal{O}_b^\dagger \mathcal{B} \rangle \\
 &> \sum_{i,j} \mathcal{G}^{ij} \langle \mathcal{A}^\dagger \text{tr}(\tau_i \Psi) \rangle \langle \text{tr}(\tau_j \Psi^\dagger) \mathcal{B} \rangle \\
 &= \sum_i \langle \mathcal{A}^\dagger \text{tr}(\tau_i \Psi) \rangle \langle \xi(\tau_i, \Psi^\dagger) \mathcal{B} \rangle
 \end{aligned} \tag{5.1}$$

We have truncated the sum over operators of the $SU(N)$ theory to those half-BPS operators made from a single complex scalar Ψ . In the sum i and j range over the conjugacy classes of C_n . We have used the fact that the inverse of the metric on the trace basis is the correlator of the dual basis $\mathcal{G}^{ij} = \langle \xi(\tau_i, \Psi^\dagger) \xi(\tau_j, \Psi) \rangle$, which effects the change of basis from the trace basis to the dual basis (3.8). If we set $\mathcal{B} = \mathcal{A}$ and divide both sides of (5.1) by $\langle \mathcal{A}^\dagger \mathcal{A} \rangle$ we get a sum of well-defined, positive probabilities

$$P(\mathcal{A} \rightarrow \text{tr}(\tau_i \Psi)) = P(\mathcal{A} \rightarrow \xi(\tau_i, \Psi)) = \frac{\langle \mathcal{A}^\dagger \text{tr}(\tau_i \Psi) \rangle \langle \xi(\tau_i, \Psi^\dagger) \mathcal{A} \rangle}{\langle \mathcal{A}^\dagger \mathcal{A} \rangle} \tag{5.2}$$

If one of \mathcal{A} and \mathcal{B} is a polynomial in Ψ then we can connect the $SU(N)$ factorisation (5.1) to the $U(N)$ factorisation. The first step is to use $\xi(\tau_i, \Psi) = \xi(\tau_i, \Phi)$

$$\sum_i \langle \mathcal{A}(\Psi^\dagger) \text{tr}(\tau_i \Psi) \rangle \langle \xi(\tau_i, \Psi^\dagger) \mathcal{B} \rangle = \sum_i \langle \mathcal{A}(\Psi^\dagger) \text{tr}(\tau_i \Psi) \rangle \langle \xi(\tau_i, \Phi^\dagger) \mathcal{B} \rangle \tag{5.3}$$

Because $\text{tr}(\Psi) = 0$ we can add back in the conjugacy classes of S_n with 1-cycles since these terms are zero

$$\sum_i \langle \mathcal{A}(\Psi^\dagger) \text{tr}(\tau_i \Psi) \rangle \langle \xi(\tau_i, \Phi^\dagger) \mathcal{B} \rangle = \sum_I \langle \mathcal{A}(\Psi^\dagger) \text{tr}(\sigma_I \Psi) \rangle \langle \xi(\sigma_I, \Phi^\dagger) \mathcal{B} \rangle \tag{5.4}$$

Here I ranges over the conjugacy classes of S_n . Finally we use $\langle \Psi^\dagger \Psi \rangle = \langle \Psi^\dagger \Phi \rangle$ to see that $\langle \mathcal{A}(\Psi^\dagger) \text{tr}(\tau_i \Psi) \rangle = \langle \mathcal{A}(\Psi^\dagger) \text{tr}(\tau_i \Phi) \rangle$ and hence

$$\sum_I \langle \mathcal{A}(\Psi^\dagger) \text{tr}(\sigma_I \Psi) \rangle \langle \xi(\sigma_I, \Phi^\dagger) \mathcal{B} \rangle = \sum_I \langle \mathcal{A}(\Psi^\dagger) \text{tr}(\sigma_I \Phi) \rangle \langle \xi(\sigma_I, \Phi^\dagger) \mathcal{B} \rangle \tag{5.5}$$

This is now a sum over $U(N)$ operators, which gives us the $U(N)$ factorisation. This only works if one of \mathcal{A} and \mathcal{B} is a function of Ψ . If σ_I contains 1-cycles the summand vanishes because $\langle \Psi^\dagger \text{tr}(\Phi) \rangle = 0$. So what we are really saying is that if one of \mathcal{A} and \mathcal{B} is a polynomial in $\Psi = \Phi - \text{tr}(\Phi)/N$ we can truncate the $U(N)$ factorisation (5.5) to the $SU(N)$ factorisation (5.1). If we translate this into probabilities it means that $P(\mathcal{A}(\Psi) \rightarrow \text{tr}(\tau_i \Psi)) = P(\mathcal{A}(\Psi) \rightarrow \text{tr}(\tau_i \Phi))$.

Since $\xi(\tau_j, \Psi) = \xi(\tau_j, \Phi)$ is a polynomial in Ψ we find the probability

$$P(\xi(\tau_j, \Psi) \rightarrow \text{tr}(\tau_i \Psi)) = \delta_{ij} \tag{5.6}$$

which is exactly the same as the corresponding $U(N)$ result $P(\xi(\tau_j, \Phi) \rightarrow \text{tr}(\tau_i \Phi))$.

For a transition into two separate states we use the factorisation on a 4-dimensional ‘genus one’ surface

$$\begin{aligned} \langle \mathcal{A}^\dagger \mathcal{B} \rangle_{G=1} &> \sum_{i,j} \sum_{k,l} \mathcal{G}^{ij} \mathcal{G}^{kl} \langle \mathcal{A}^\dagger \text{tr}(\tau_i \Psi) \text{tr}(\tau_k \Psi) \rangle \langle \text{tr}(\tau_l \Psi^\dagger) \text{tr}(\tau_j \Psi^\dagger) \mathcal{B} \rangle \\ &= \sum_i \sum_k \langle \mathcal{A}^\dagger \text{tr}(\tau_i \Psi) \text{tr}(\tau_k \Psi) \rangle \langle \xi(\tau_k, \Psi^\dagger) \xi(\tau_i, \Psi^\dagger) \mathcal{B} \rangle \end{aligned} \quad (5.7)$$

If one of \mathcal{A} and \mathcal{B} is a function of Ψ then the $U(N)$ factorisation truncates to this result.

The probability of a transition to KK gravitons is given by

$$P(\mathcal{A} \rightarrow \text{tr}(\tau_i \Psi), \text{tr}(\tau_k \Psi)) = \frac{\langle \mathcal{A}^\dagger \text{tr}(\tau_i \Psi) \text{tr}(\tau_k \Psi) \rangle \langle \xi(\tau_k, \Psi^\dagger) \xi(\tau_i, \Psi^\dagger) \mathcal{A} \rangle}{\langle \mathcal{A}^\dagger \mathcal{A} \rangle_{G=1}} \quad (5.8)$$

For $\mathcal{A} = \xi(\tau_m, \Psi)$

$$P(\xi(\tau_m, \Psi) \rightarrow \text{tr}(\tau_i \Psi), \text{tr}(\tau_k \Psi)) \quad (5.9)$$

$$\begin{aligned} &= \frac{\langle \xi(\tau_m, \Psi^\dagger) \text{tr}(\tau_i \Psi) \text{tr}(\tau_k \Psi) \rangle \langle \xi(\tau_k, \Psi^\dagger) \xi(\tau_i \Psi^\dagger) \xi(\tau_m, \Psi) \rangle}{\langle \xi(\tau_m, \Psi^\dagger) \xi(\tau_m, \Psi) \rangle_{G=1}} \\ &= \frac{\delta_{[\tau_m]=[\tau_i \circ \tau_k]} \langle \xi(\tau_k, \Psi^\dagger) \xi(\tau_i \Psi^\dagger) \xi(\tau_m, \Psi) \rangle}{\langle \xi(\tau_m, \Psi^\dagger) \xi(\tau_m, \Psi) \rangle_{G=1}} \end{aligned} \quad (5.10)$$

So $\mathcal{A} = \xi(\tau_m, \Psi)$ will decay into two multi-trace operators as long as $\tau_i \circ \tau_k$ is in the conjugacy class of τ_m .

6. Diagonalisation by higher Hamiltonians

In this section we will find a diagonal basis for the $SU(N)$ correlator.⁴ We can reduce the half-BPS sector of the $\mathcal{N} = 4$ SYM to matrix quantum mechanics [3, 5]. For gauge group $U(N)$ the Schur polynomials are eigenstates of commuting higher Hamiltonians (for $U(N)$ these correspond to the Casimirs of the Lie algebra). Our strategy will be to find eigenstates of the higher Hamiltonians for $SU(N)$. These eigenstates are necessarily diagonal.

If we do a reduction of the $\mathcal{N} = 4$ SYM action on S^3 with only the first two real scalars X_1 and X_2 turned on then we get a (0+1)-dimensional matrix model

$$S = \int dt \text{Tr} (\dot{X}_1^2 + \dot{X}_2^2 - X_1^2 - X_2^2). \quad (6.1)$$

The potential term couples to the curvature of S^3 but we have rescaled the fields appropriately. If we introduce the complex chiral scalar $Z = X_1 + iX_2$ and find its momentum conjugate Π then we can define harmonic oscillator operators $A = Z + i\Pi$ and $B = Z - i\Pi$ and their conjugates A^\dagger and B^\dagger . These satisfy standard commutation relations

$$[A_a, A_b^\dagger] = g_{ab} \quad (6.2)$$

⁴This section was done in collaboration with Sanjaye Ramgoolam.

where a, b run over the adjoint representation of the gauge group and g_{ab} is the inverse of the bilinear invariant form $g^{ab} = \text{tr}(T^a T^b)$.

Our Hamiltonian is

$$H = \text{tr}(A^\dagger A + B^\dagger B) \tag{6.3}$$

and our angular momentum operator is

$$J = \text{tr}(A^\dagger A - B^\dagger B) \tag{6.4}$$

For $\text{tr}((A^\dagger)^n (B^\dagger)^m) |0\rangle$, $E = n + m$, $J = n - m$. For our highest weight chiral primaries we have $E = J$ so m is zero and we restrict to the $\text{tr}((A^\dagger)^n) |0\rangle$ states. We have higher Hamiltonians

$$H_n = \text{tr}((A^\dagger A)^n) \tag{6.5}$$

that commute with $H = \text{tr}(A^\dagger A)$.

If we concentrate on the $U(N)$ case we find that in terms of adjoint matrix indices

$$[A_j^i, A_l^{\dagger k}] = [A_a, A_b^\dagger] (T^a)_j^i (T^b)_l^k = g_{ab} (T^a)_j^i (T^b)_l^k = \delta_l^i \delta_j^k \tag{6.6}$$

The Schur polynomials are simultaneous eigenstates of these higher Hamiltonians and the different eigenvalues give a complete identification of each Schur polynomial

$$H_n \chi_R(A^\dagger) |0\rangle = C_n^R \chi_R(A^\dagger) |0\rangle \tag{6.7}$$

For $U(N)$ these higher Hamiltonians are in fact the Casimirs of the Lie algebra (cf. [12]).

We can show that these Schur polynomials are diagonal. Suppose we make no assumptions about the correlator of the Schur polynomials and call it $h_{RS} := \langle 0 | \chi_R(A) \chi_S(A^\dagger) | 0 \rangle$.

$$\langle 0 | \chi_R(A) H_n \chi_S(A^\dagger) | 0 \rangle = C_n^S h_{RS} = C_n^R h_{RS} \tag{6.8}$$

We have acted to the right with H_n and then to the left. If $h_{RS} \neq 0$ then we must have $C_n^R = C_n^S$ for all n ; otherwise $h_{RS} = 0$. We have enough Casimirs to distinguish between the Schur polynomials so if $R \neq S$ then $C_n^R \neq C_n^S$ for some n , so we must have $h_{RS} = 0$ for $R \neq S$.

Now extend this argument to the $SU(N)$ case for which

$$[A_j^i, A_l^{\dagger k}] = g_{ab} (T^a)_j^i (T^b)_l^k = \delta_l^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_l^k \tag{6.9}$$

The higher Hamiltonians no longer have simple eigenvectors or eigenvalues. Also the higher Hamiltonians no longer correspond to the Casimirs of $SU(N)$. However they must diagonalise the correlator by the same argument as above.

For example, at level 4 we have two independent gauge-invariant states for which

$$\begin{aligned} H \text{tr}(A^{\dagger 2}) \text{tr}(A^{\dagger 2}) |0\rangle &= 4 \text{tr}(A^{\dagger 2}) \text{tr}(A^{\dagger 2}) |0\rangle \\ H \text{tr}(A^{\dagger 4}) |0\rangle &= 4 \text{tr}(A^{\dagger 4}) |0\rangle \\ H_2 \text{tr}(A^{\dagger 2}) \text{tr}(A^{\dagger 2}) |0\rangle &= \left[\left(4N - \frac{8}{N} \right) \text{tr}(A^{\dagger 2}) \text{tr}(A^{\dagger 2}) + 8 \text{tr}(A^{\dagger 4}) \right] |0\rangle \\ H_2 \text{tr}(A^{\dagger 4}) |0\rangle &= \left[\left(4 + \frac{12}{N^2} \right) \text{tr}(A^{\dagger 2}) \text{tr}(A^{\dagger 2}) + \left(4N - \frac{28}{N} \right) \text{tr}(A^{\dagger 4}) \right] |0\rangle \end{aligned} \tag{6.10}$$

If we find the eigenvectors of H_2 , we get a diagonal basis

$$\left[\left(\frac{5}{4N} - \frac{\sqrt{49N^2 + 8N^4}}{4N^2} \right) \text{tr}(A^{\dagger 2}) \text{tr}(A^{\dagger 2}) + \text{tr}(A^{\dagger 4}) \right] |0\rangle \quad (6.11)$$

$$\left[\left(\frac{5}{4N} + \frac{\sqrt{49N^2 + 8N^4}}{4N^2} \right) \text{tr}(A^{\dagger 2}) \text{tr}(A^{\dagger 2}) + \text{tr}(A^{\dagger 4}) \right] |0\rangle. \quad (6.12)$$

This method of using eigenvectors of higher Hamiltonians to diagonalise the correlator will work at all levels. While it is as complicated as a Gram-Schmidt diagonalisation, it does at least share its derivation from the higher Hamiltonians with the $U(N)$ matrix model.

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A. Useful identities

There are two orthogonality relations for the characters of the symmetric group S_n .

1. For $\rho, \tau \in S_n$ if we sum over representations R of S_n

$$\sum_{R(n)} \chi_R(\rho) \chi_R(\tau) = \frac{n!}{|[\tau]|} \delta([\tau] = [\rho]) \quad (A.1)$$

This generalises for $\rho_i \in S_{n_i}$ and $\tau \in S_n$ where $n = \sum_i n_i$ and $i = 1, 2, \dots, k$

$$\begin{aligned} \sum_{R_1(n_1), R_2(n_2), \dots, R_k(n_k), T(n)} g(R_1, R_2, \dots, R_k; T) \chi_{R_1}(\rho_1) \chi_{R_2}(\rho_2) \cdots \chi_{R_k}(\rho_k) \chi_T(\tau) \\ = \frac{n!}{|[\tau]|} \delta([\tau] = [\rho_1 \circ \rho_2 \cdots \circ \rho_k]) \end{aligned} \quad (A.2)$$

2. For representations R, S of S_n if we sum over the elements of S_n

$$\sum_{\sigma \in S_n} \chi_R(\sigma) \chi_S(\sigma) = n! \delta_{RS} \quad (A.3)$$

This generalises for representations R_i of S_{n_i}

$$\begin{aligned} \frac{1}{n_{R_1}! \cdots n_{R_k}!} \sum_{\rho_1 \in S_{n_1}} \cdots \sum_{\rho_k \in S_{n_k}} \chi_{R_1}(\rho_1) \cdots \chi_{R_k}(\rho_k) \chi_T(\rho_1 \circ \cdots \circ \rho_k) \\ = g(R_1, \dots, R_k; T) \end{aligned} \quad (A.4)$$

which is the Littlewood-Richardson coefficient.

We can also derive from (A.4)

$$\chi_T(\rho_1 \circ \cdots \circ \rho_k) = \sum_{R_1(n_1)} \cdots \sum_{R_k(n_k)} g(R_1, \dots, R_k; T) \chi_{R_1}(\rho_1) \cdots \chi_{R_k}(\rho_k) \quad (A.5)$$

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